§ 11 POLARIZATIONS

Notiztitel

Version 1.2

In quantum mechanics one can represent the Hilbert space of states as the space of square integrable complex functions on the spectrum of a given complete set of commuting observables. In the classical situation a natural choice of a "complete set of commuting observables" on a given symplectic manifold (M,ω) of dimension $2n$ is a set of $n = \frac{1}{2}\dim M$ functions $F_1, F_2, \dots F_n \in \mathcal{E}(M,\mathbb{R})$, which are independent at each point of M, commuting in the sense of

$$\{F_i, F_j\} = 0 \qquad \text{for all } i,j \in \{1,\dots 2\},$$

and such that the corresponding Hamiltonian vector fields $X_{F_1}, X_{F_2}, \dots X_{F_n}$ are all complete. We thus arrive at the notion of a completely integrable system.

In general, however, such classical observables do not exist. As a result, one has to relax the condition of globally defined F_i and one considers instead such concepts as the notion of a distribution, a foliation or a polarization. Moreover, one generalizes from $\mathbb{R}$-valued observables to $\mathbb{C}$-valued observables and therefore studies complex polarizations.

A. Distributions:

(11.1) DEFINITION: A DISTRIBUTION D on a manifold M is a subbundle D of the tangent bundle TM: $D \subset TM$.

A distribution is therefore given by a linear subspace $D_a \subset T_aM$ over $\mathbb{R}$ at each point $a \in M$ such that there exists for each point $a \in M$ an open neighborhood $\mathcal{U}$ and k (smooth) vector fields $X_1, \dots X_k \in \mathcal{V}(\mathcal{U})$ with

$$(*) \quad D_b = \operatorname{span}\{ X_j(b) \mid 1 \leq j \leq k \}, \quad b \in \mathcal{U}.$$

In this description the natural number k can be chosen to be the dimension or rank of the vector bundle D. If the distribution D shall be given by the D_a, $a \in M$, then - in addition to $(*)$ - one has to require $\dim_{\mathbb{R}} D_a =$ constant; or, equivalently, that k is always minimally chosen.

(11.2) DEFINITION: A distribution $D \subset TM$ is called INTEGRABLE if for each $a \in M$ there exists a k-dimensional submanifold N in an open neighbourhood $\mathcal{U}$ of a so that

1° $a \in M$,

2° for each $b \in N$: $T_bN = D_b$.

Any submanifold $N \subset \mathcal{U}$ with 2° is called an INTEGRAL MANIFOLD of the distribution.

Distributions $D \subset TM$ with $\dim_{\mathbb{R}} D_a = 1 = \mathrm{rk}\, D$ are always integrable, since locally $D_a = \mathbb{R}X(a)$, $a \in \mathcal{U}$, for a (local) nowhere vanishing vector field X, and the N_a are given by the integral curves of X.

Without proof we state the fundamental theorem of Frobenius.

(11.3) PROPOSITION (FROBENIUS) A necessary and sufficient condition for a distribution to be integrable is that the global sections of D form a Lie-subalgebra of $\mathcal{V}(M)$:

$$X, Y \in \Gamma(M, D) \Rightarrow [X, Y] \in \Gamma(M, D).$$

An integrable distribution is also called a FOLIATION. The maximal connected integral manifolds, i.e. the $N \subset M$ with $T_b N = D_b$ for all $b \in N$ are called the LEAVES of the foliation. Let M/D be the space of leaves.

(11.4) DEFINITION: A foliation is called REDUCIBLE (or ADMISSIBLE) if M/D exists as a quotient manifold and the canonical map $\pi : M \to M/D$ is a submersion.

The last condition is equivalent to π having maximal rank at all points of M.

(11.5) EXAMPLES: 1° The vertical distribution: Let $Q \subset \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$ with the standard coordinates $q^1, q^2, \dots q^n$. The cotangent bundle $T^*Q \cong Q \times \mathbb{R}^n$ with standard (and canonical with respect to $\omega = dq^k \wedge dp_k = -d\lambda$) coordinates $q^1, \dots q^n, p_1, \dots p_n$ has the VERTICAL distribution D spanned by

$$\left\{ \frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n} \right\}, \quad D \subset T(T^*Q).$$

D is integrable and the leaves are given by

$$\left\{ (q,p) \in Q \times \mathbb{R}^n : q = c \right\} = \{c\} \times \mathbb{R}^n = \tau^{-1}(c)$$

$c \in Q$ constant ($\tau : T^*Q \to Q$ is the projection). Moreover, $T^*Q / D \cong Q$, and D is reducible.

In this example the distribution D is generated by the globally defined functions $q^1 \dots q^n \in \mathcal{E}(T^*Q, \mathbb{R})$ in involution $\left(\{q^j, q^k\} = 0 \right)$: $X_{q^k} = \frac{\partial}{\partial p^k}$ and

$$D = \text{span} \left\{ X_{q^k} ; \; k = 1, 2, \dots n \right\}$$

The cotangent bundle $M = T^*Q$ with respect to a general manifold Q of dimension has again the vertical distribution D given locally by $\frac{\partial}{\partial p_1}, \dots \frac{\partial}{\partial p_n}$ in suitable canonical bundle charts (q,p) and leaves $\tau^{-1}(c)$ $c \in Q$. We do not have global $F_1, \dots F_n$ which Poisson-commute and whose Hamiltonian vector fields X_{F_k} generate D, in general.

2° (Horizontal distribution) $Q \subset \mathbb{R}^n$ open and $M = T^*Q$ with D given by $\left\{ \frac{\partial}{\partial q^1}, \dots \frac{\partial}{\partial q^n} \right\}$. The leaves are

$$\{ (q,p) \in \mathbb{R}^n \times \mathbb{R}^n \mid p = c \} = Q \times \{c\}, \quad c \in \mathbb{R}^n;$$

and $M/D \cong \mathbb{R}^n$.

3° (Radial distribution). $M = \mathbb{R}^2 \setminus \{0\}$ with coordinates (q,p). Let D be the distribution generated by

$$q \frac{\partial}{\partial p} - p \frac{\partial}{\partial q}.$$

D is a distribution with leaves $\{ p^2 + q^2 = r^2 \}$, $r \in \mathbb{R}$, $r > 0$. $M/D \cong \mathbb{R}_+ =]0, \infty[$. So D is integrable.

Note that the vertical distribution D on $M = \mathbb{R}^2 \setminus \{0\} \subset \mathbb{R}^2$ induced by the projection $pr_1 : M \to \mathbb{R}$ is not reducible.[*] The leaves are the vertical lines through $(0,p)$, $p \neq 0$, and the two rays

$$\{(q,0) : q > 0\} \text{ and } \{(q,0) : q < 1\}.$$

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[*] Check!

B. Real polarizations:

(11.6) DEFINITION: Let (M,ω) be a symplectic manifold. A REAL POLARIZATION on M is a foliation (i.e. an integrable distribution) $D \subset TM$ on M which is MAXIMALLY ISOTROPIC, i.e. for all $a \in M$

$$\omega_a(X,Y) = 0 \quad \text{for } X,Y \in D_a,$$

and no larger subspace of T_aM which contains D_a properly has this property. A real polarization is called REDUCIBLE (or ADMISSIBLE) if it is reducible as a distribution.

We see that the three examples in 11.5 are reducible real polarizations.

(11.7) PROPOSITION: Let (M,ω) be a symplectic manifold of dimension $2n$. Then a given (smooth) distribution $D \subset TM$ is a real polarization if and only if for each $a \in M$ there exists an open neighbourhood U of a and n independent smooth functions $F_1, \dots, F_n \in \mathcal{E}(U,\mathbb{R})$ such that

1° For each $a \in U$ $D_a = \mathrm{span}_{\mathbb{R}}\{X_{F_1}(a), \dots, X_{F_n}(a)\}$

2° $\{F_j, F_k\} = 0$ (i.e. $\omega|_U(X_{F_j}, X_{F_k}) = 0$) for all $j,k \in \{1,2,\dots n\}$.

□ Proof. Let D be a real polarization. Since D is, in particular, an integrable distribution there exists locally n independent smooth functions $F_1, .. F_n \in \mathcal{E}(U,\mathbb{R})$

such that the leaves of D are locally of the form

$$\{a \in \mathcal{U} : F_1(a) = c_1, \dots F_n(a) = c_n\}$$

with suitable constants $c_1, \dots c_n \in \mathbb{R}$. For each vector field $X \in \Gamma(\mathcal{U}, D)$ we have

$$X(F_i) = 0 \,, \quad i = 1, 2, \dots n,$$

hence,

$$\omega(X_{F_i}, X) = dF_i(X) = X(F_i) = 0$$

It follows $X_{F_i} \in \Gamma(\mathcal{U}, D)$, since D is maximally isotropic, and therefore, $X_{F_1}, \dots X_{F_n}$ span D locally (note, that the $F_1, \dots F_n$ are independent, i.e. $dF_1(a), \dots dF_n(a)$ are linearly independent for each $a \in \mathcal{U}$). As a consequence, by isotropy, we have $\omega(X_{F_i}, X_{F_j}) = 0$, hence

$$\{F_i, F_j\} = 0 .$$

Thus, we have shown 1° and 2°.

Conversely, the condition $\{F_i, F_j\} = 0$, i.e. $\omega(X_i, X_j) = 0$ on $\mathcal{U}$ implies, that D is isotropic. Since the F_i are independent, $\dim_{\mathbb{R}} D_a = n$, and the leaves of D are given on $\mathcal{U}$ by

$$\{a \in \mathcal{U} \mid F_1(a) = c_1, \dots F_n(a) = c_n\} . \qquad \square$$

In general, real polarizations need not exist on a given symplectic manifold.

(11.8) EXAMPLE: S^2 with the symplectic form ω the volume form given by $\frac{dx_1 \wedge dx_2}{x_3}$ does not admit a real polarization $D \subset TS^2$.
$D \subset TS^2$ would be a 1-dimensional real bundle whose leaves are 1-dimensional compact submanifolds $S \subset S^2$. As a result, TS^2 would have a global non-vanishing section in contradiction to the "Satz vom Igel".

For the purpose of geometric quantization we thus need a generalization of the notion of a real polarization: The complex polarization!

C. The complex linear case:

Before giving the definition of a complex polarization, we study the linear case, i.e. we consider a $2n$ dimensional symplectic vector space (V, ω) (as the prototype of the tangent space T_aM, $a \in M$, of a symplectic manifold (M, ω)).

Let $V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C} = V \oplus iV$ its complexification with the obvious extension of $\omega : V \times V \to \mathbb{R}$ to $V^{\mathbb{C}}$:

$$\omega : V^{\mathbb{C}} \times V^{\mathbb{C}} \to \mathbb{C},$$

$$\omega(v+iw, v'+iw') := \omega(v,v') - \omega(w,w') + i(\omega(v,w') + \omega(w,v')).$$

Let $P \subset V^{\mathbb{C}}$ a complex Lagrangian subspace, i.e. P

is a complex linear subspace, P is isotropic (for all $Z, W \in P$: $\omega(Z,W) = 0$) and maximally isotropic (if $Q \subset V^{\mathbb{C}}$ is a complex isotropic subspace with $P \subset Q$ it follows $P = Q$). ω defines a hermitian form on P

$$\langle Z, W\rangle := -2i\omega(Z, \bar{W}) \qquad \left(\overline{v+iz} := v - iz,\ v, z \in V\right)$$

Let N be the null space of $\langle\, ,\rangle$:

$$N = \{ Z \in P \mid \langle Z, W\rangle = 0 \text{ for all } W \in P\} = P^{\perp}.$$

It is easy to show $N = P \cap \bar{P}$! [*]
Therefore, $\langle\, ,\rangle$ projects to a nondegenerate form of signature (r,s) on P/N, i.e. its matrix is

$$\operatorname{diag}(\underbrace{1, 1, \ldots 1}_{r}, \underbrace{-1, \ldots -1}_{s}), \quad 0 \leq r+s = n - \dim_{\mathbb{C}} N,$$

with respect to a suitable basis of P/N.

P is said of TYPE (r,s).
In case of $s = 0$, P is called POSITIVE.
In case of $r = s = 0$, P is called REAL and we have $P = D^{\mathbb{C}}$ for a real Lagrangian subspace $D \subset V$.
In case of $r + s = n$, i.e. $N = P \cap \bar{P} = \{0\}$, the complex Lagrangian subspace P is called KÄHLER (or PURELY COMPLEX).

The real subspaces

$$D := P \cap \bar{P} \cap V \quad \text{and} \quad E = (P + \bar{P}) \cap V$$

are of special interest in the following.

[*] Übung: Check!

Note, that $D^{\mathbb{C}} = P \cap \overline{P}$ and $E^{\mathbb{C}} = P + \overline{P}$.

The number $n-(r+s) = \dim_{\mathbb{C}} P \cap \overline{P} = \dim_{\mathbb{R}} D$ is sometimes called the number of "real directions" in P.

We can find $\mathbb{R}$-linear independent vectors $a_1, \dots a_n$ and $b_1, \dots b_n$ in V, such that

$$D = \mathrm{span}_{\mathbb{R}} \{a_1, \dots a_k\}, \quad D^{\mathbb{C}} = P \cap \overline{P} = N, \quad \dim_{\mathbb{C}} N = k$$

$$P = \mathrm{span}_{\mathbb{C}} \{a_1, \dots a_k\} \cup \{ a_j + i b_j \mid k < j \leq n \},$$

$$\overline{P} = \mathrm{span}_{\mathbb{C}} \{a_1, \dots a_k\} \cup \{ a_j - i b_j \mid k < j \leq n \},$$

$$E = \mathrm{span}_{\mathbb{R}} \{a_1, \dots a_k\} \cup \{ a_j \pm i b_j \mid k < j \leq n \}, \quad E^{\mathbb{C}} = P + \overline{P},$$

such that the symplectic form satisfies

$$\omega(a_j, a_l) = \omega(b_j, b_l) = 0 \qquad \text{for all } 1 \leq j, l \leq n,$$

$$\omega(a_j, b_l) = 0 \qquad \text{for all } 1 \leq j \leq k < l \leq n,$$

in order to have $N = D^{\mathbb{C}}$. Moreover, the basis $\{a_j, b_l\}$ of V (over $\mathbb{R}$) can be chosen with

$$\omega(a_j, b_l) = \omega(a_l, b_j) = +\tfrac{1}{4} i \, \delta_{jl}, \quad k < j \leq k+r, \; 1 \leq l \leq n$$

$$\omega(a_j, b_l) = \omega(a_l, b_j) = -\tfrac{1}{4} i \, \delta_{jl}, \quad k+r < j \leq n, \; 1 \leq l \leq n$$

in order to obtain the above mentioned form $\langle \, , \rangle$

on P/N by the matrix

$$\mathrm{diag}\,(\underbrace{1, \ldots\, 1}_{r}, \underbrace{-1, \ldots\, -1}_{s}) \qquad , \; k+r+s = n.$$

In the Kähler case, $P \cap \overline{P} = \{0\}$ and $V^{\mathbb{C}} = P \oplus \overline{P}$ (and $(a_1, \ldots a_n, b_1, \ldots b_n)$ is a base of V).
Any $v \in V$ is of the form

$$v = z + \overline{z} \quad \text{with a unique } z = z(v) \in P.$$

Indeed, $v = z \oplus w$ with $z \in P$, $w \in \overline{P}$, and $v = \overline{v}$,
i.e. $z \oplus w = \overline{w} \oplus \overline{z}$ with $z = \overline{w}$, $w = \overline{z}$.

The map $J : V \to V$

$$Jv := i z(v) - i \overline{z(v)}$$

is $\mathbb{R}$-linear and satisfies $J^2 = -1 = -\,\mathrm{id}$.

Any $\mathbb{R}$-linear map $J : V \to V$ with $J^2 = -1$ is called an ALMOST COMPLEX STRUCTURE. It is easy to show that if J is an almost complex structure then by

$$(\alpha + i\beta)(v) := \alpha v + \beta J(v) \quad , v \in V, \; \alpha, \beta \in \mathbb{R},$$

a scalar multiplication $\mathbb{C} \times V \to V$ is defined making V into a complex vector space.

Conversely, if V is the underlying real space of a complex vector space, then

$$v \longmapsto iv \qquad (\text{or } v \longmapsto -iv)$$

is an almost complex structure. Thus, on a real vector space V an almost complex structure is essentially the same as a complex structure. (This is no longer true for manifolds, where a complex structure given by holomorphically compatible charts induces almost complex structures in the tangent spaces T_aM, but the converse need not hold. There are manifolds M with a tensor field $J \in \Gamma(M, \mathrm{End}(TM))$ with $J^2 = -\mathrm{id}$, such that M is not a complex manifold. Cf. 12.1.)

In our situation of V with Kähler polarization[(+)] P and J given by $Jv = iZ(v) - i\overline{Z(v)}$ we have

$$\omega(Jv, Jw) = \omega(v,w), \quad v,w \in V,$$

i.e. J is (by definition) COMPATIBLE with ω.

Conversely, given an almost complex structure J on V which is compatible with ω we obtain a Kähler polarization[(+)] $P := \{(J+i)z : z \in V^{\mathbb{C}}\}$ (J has to be extended complex linearly to $V^{\mathbb{C}}$ and iz is the multiplication of i with z in $V^{\mathbb{C}}$).

As a result, there is a natural bijective correspondence between the set of Kähler polarizations[(+)] on $V^{\mathbb{C}}$ and the set of compatible almost complex structures on (V, ω).

(+) i.e. a complex Lagrangian subspace $P \subset V^{\mathbb{C}}$.

D. Complex polarizations:

We now come to the definition of general polarizations.

(11.9) DEFINITION: Let (M,ω) be a symplectic manifold of dimension $2n$. A (COMPLEX) POLARIZATION P of (M,ω) is a complex subbundle $P \subset TM^{\mathbb{C}}$ of complex dimension n such that

1° For $X,Y \in \Gamma(M,P)$ we have $[X,Y] \in \Gamma(M,P)$ (P is INVOLUTIVE)

2° P is Lagrangian: $P_a \subset T_aM^{\mathbb{C}}$ is maximally isotropic for all $a \in M$.

3° $D_a := P_a \cap \overline{P}_a \cap T_aM$ has constant rank $k \in \{0,1,..n\}$.

P is called to be

REAL if $D_a = T_aM$ for all $a \in M$,

KÄHLER if $D_a = \{0\}$ " " " , cf. subsection C

POSITIVE if $s = 0$ (see above)

(sometimes "Kähler polarization" means $D = 0$ & positive)

STRONGLY INTEGRABLE if the distribution $E \subset TM$ ($E_a := (P_a + \overline{P}_a) \cap T_aM$) is integrable. (Not that D is integrable by 1°.)

TYPE (r,s) if P_a is of type (r,s) for all $a \in M$

Our examples above (11.5) are of type $(0,0)$.

A typical Kähler polarization is the following (special case of the general description in subsection C):

(11.10) EXAMPLES: 1° $M = T^*\mathbb{R}^n$ with the standard symplectic form $\omega = dq^j \wedge dp_j$. Let $z_j := p_j + i q^j$ and

$$P_a := \mathbb{C}\frac{\partial}{\partial z_1} + \dots + \mathbb{C}\frac{\partial}{\partial z_n} = \operatorname{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial z_j} : 1 \le j \le n\right\}$$

The distribution $P \subset TM^{\mathbb{C}}$ defined by P_a, $a \in M$, is involutive: For $Z = \xi^j \frac{\partial}{\partial z_j}$, $W = w^j \frac{\partial}{\partial z_j}$ we have

$$\begin{aligned}
[Z, W] f &= \xi^j \frac{\partial}{\partial z_j}\left(w^k \frac{\partial}{\partial z_k} f\right) - w^k \frac{\partial}{\partial z_k}\left(\xi^j \frac{\partial}{\partial z_j} f\right) \\
&= \xi^j \frac{\partial w^k}{\partial z_j} \frac{\partial}{\partial z_k} f - w^k \frac{\partial \xi^j}{\partial z_k} \frac{\partial}{\partial z_j} f \qquad \left(\frac{\partial}{\partial z_j}\frac{\partial}{\partial z_k} f = \frac{\partial}{\partial z_k}\frac{\partial}{\partial z_j}\right) \\
&= \left(\xi^j \frac{\partial w^k}{\partial z_j} - w^j \frac{\partial \xi^k}{\partial z_j}\right) \frac{\partial}{\partial z_k} f .
\end{aligned}$$

We have
$$\begin{aligned}
dz_j \wedge d\bar{z}_j &= d(p_j + i q^j) \wedge d(p_j - i q^j) \\
&= i\, dq^j \wedge dp_j - i\, dp_j \wedge dq^j = 2i\, dq^j \wedge dp_j
\end{aligned}$$

Hence, our standard symplectic form $\omega = dq^j \wedge dp_j$ can be written as

$$\omega = \frac{1}{2i} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j = -\frac{1}{2i} \sum_{j=1}^{n} d\bar{z}_j \wedge dz_j$$

Using this form of ω we immediately conclude $\omega\left(\frac{\partial}{\partial z_k}, \frac{\partial}{\partial z_\ell}\right) = 0$, i.e. P is Lagrangian. (Note that

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial p_j} - i \frac{\partial}{\partial q^j}\right), \quad \frac{\partial}{\partial \bar{z}_j} = \overline{\frac{\partial}{\partial z_j}} = \frac{1}{2}\left(\frac{\partial}{\partial p_j} + i \frac{\partial}{\partial q^j}\right).)$$

Since, moreover

$$\overline{P}_a := \mathrm{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \bar{z}_j} : 1 \leq j \leq n \right\},$$

it follows that

$$P_a \cap \overline{P}_a = \{0\} \quad \& \quad P_a + \overline{P}_a = T_a(T^*M),$$

Hence, P_a defines a Kähler polarization P,
Note, that $\overline{P}_a$ defines a Kähler polarization as well.

2° More generally, any Kähler manifold (M, ω) has a natural Kähler polarization, the so called holomorphic polarization.

A Kähler manifold is a complex manifold M of complex dimension n which carries a symplectic form ω, such that the complex structure and the symplectic structure are compatible in the following sense:

The complex structure defines a distribution $P \subset TM^{\mathbb{C}}$ given by

$$P_a = \mathrm{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z_j} \,\middle|\, 1 \leq j \leq n \right\}, \quad a \in M,$$

where $z_1, \ldots z_n$ are holomorphic coordinates in a neighbourhood of a. (For a holomorphic chart $\varphi : U \to V \subset \mathbb{C}^n$ in a neighbourhood U of a we have the coordinates $z_j = \varphi_j$, $\varphi = (\varphi_1, \varphi_2, \dots \varphi_n)$, $z_j = x_j + iy_j$. And $\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$.) P_a is independent of the coordinates since for another set of holomorphic coordinates we have

$$\frac{\partial}{\partial z_j} = \frac{\partial w_k}{\partial z_j} \cdot \frac{\partial}{\partial w_k}, \text{ with } \left(\frac{\partial w_k}{\partial z_j}\right) \in GL(n, \mathbb{C}).$$

The 2 structures are compatible if P_a is isotropic (hence maximally isotropic). Therefore, $P \subset TM^{\mathbb{C}}$ is a complex Lagrangian distribution. P is involutive as can be shown in the same way as in 1°. Finally, also

$$P \cap \overline{P} = \{0\} \quad , \quad P + \overline{P} = TM^{\mathbb{C}}$$

follows as above.

We have presented the definition of a Kähler manifold in the spirit of polarizations. But we have to require one more property for (M, ω) to be a Kähler manifold: $-i\omega$ has to be positive definite (on the holomorphic tangent space, i.e. on P). Using the positivity of $-i\omega$ we can come to another (but equivalent) definition of a Kähler manifold:

A Kähler manifold is a complex manifold M with a Riemannian metric g such that the almost complex structure J is compatible with g:

$$g(X,Y) = g(\mathcal{J}X, \mathcal{J}Y) \quad \text{for all } X, Y \in T_aM, \ a \in M.$$

Moreover, the induced form $\omega(X,Y) := g(\mathcal{J}X, Y)$ is closed. Hence (M, ω) is, in particular a symplectic manifold.

Hence, in case of a Kähler manifold (M, ω) the holomorphic distribution P on M is a Kähler polarization which is moreover positive.

We see that 1° is simply a special case of 2°, and the work in Example 1° to be done was essentially to show that $T^*\mathbb{R}^n$ with the „complex coordinates" $z_j := p_j + iq^j$, $j = 1, \dots n$, and the standard symplectic structure is a Kähler manifold.

E. Coming back to quantization:

The purpose of introducing polarizations was to reduce the set of wave functions. This is done by restricting to those functions, sections, vector fields which respect the polarizations.

For example, given a general complex polarization on (M, ω) the polarized functions are the functions $f \in \mathcal{E}(M, \mathbb{C})$ with

$$L_X f = 0 \quad \text{for all } X \in \Gamma(M,\bar{P}) =: \mathcal{V}_{\bar{P}}(M).^{(+)}$$

And the polarized sections in a line bundle L over M with connection ∇ are the sections $s \in \Gamma(M,L)$ with

$$\nabla_X s = 0 \quad \text{for all } X \in \Gamma(M,\bar{P}).$$

With respect to a polarization $P \subset TM^{\mathbb{C}}$ we see that the Hilbert space of „wave functions" corresponding to a prequantum bundle (L,∇,H) on (M,ω) should be based on the space

$$\Gamma_P(M,L) := \{ s \in \Gamma(M,L) \mid \nabla_X s = 0 \;\; \forall X \in \mathcal{V}_{\bar{P}}(M) \}$$

of polarized sections.

More generally, we look at the sheaf $\mathcal{L}_P$ of germs of polarized sections and can consider the cohomology spaces

$$H^q(M, \mathcal{L}_P)$$

as the basic building block which should lead to the appropriate Hilbert space. $\left(H^0(M,\mathcal{L}_P) = \Gamma(M,\mathcal{L}_P) = \Gamma_P(M,L) \right)$

In general, we have to describe what kind of scalar product on $\Gamma_P(M,L)$ is reasonable. One can not simply integrate along the induced volume form on M: For example with respect to the

(+) In other texts, the condition is required for $X \in \Gamma(M,P)$!

vertical polarization on $M = T^*\mathbb{R}^n$, a polarized section s is constant on the fibres of $\tau : T^*\mathbb{R}^n \to \mathbb{R}^n$, hence $\int_M \langle s,s \rangle \, d\varepsilon = \infty$.

It is much better only to integrate along $\mathbb{R}^n$. But in the case of $M = T^*Q$ there is no natural volume form on Q.

For a real polarization P we look at the space of leaves M/P. We can hope that it carries some natural volume form ε and integrate along M/P to obtain

$$\left\{ s \in \Gamma_P(M,L) \;\middle|\; \langle s,s \rangle := \int_{M/P} \langle s,s \rangle \, d\varepsilon < \infty \right\}$$

as a suitable pre Hilbert space. Note however, that M/P is in general not a manifold and the quotient topology can be non Hausdorff. To overcome these difficulties one introduces half densities in order to get a reasonable Hilbert space out of $\Gamma_P(M,L) = \{\text{polarized section of } L\}$, where (L, ∇, H) is a prequantum bundle.

In general, one faces more difficulties and has to consider D and E. In particular, the quotient $M/_E$ is of interest ($E = (P + \bar{P}) \cap TM$) but E is in general not integrable and if so, $M/_E$ need not be a manifold.